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Metrics in the Permutation Space

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Abstract. In various problems the solution is completely specified by determining an optimal permutation of $\{1,\dots,n\}$. To use an optimum-seeking method one has to define the space of permutations, i.e., to define the metric of this space. The development of this area began with the pioneering work of E.S. Page [1]. In this paper we present two other, more efficient metrics.

INTRODUCTION

In scheduling, communication theory, integer programming problems, etc., the solution is completely specified by determining an optimal permutation of $\{1,\dots,n\}$. To use an optimum-seeking method one has first to define the space of permutations, i.e., to define the distance between the two elements of this space. Thus each space of permutations is defined by its metric.

The development of this research area began with the pioneering work of Page [1]. Page suggested his well-known chain metric (which we will henceforth denote by M_1) and used it in various scheduling problems. The distance in the chain metric is defined as the minimal number of cuts in one of the permutations necessary to compose the other one. The main shortcoming of the chain metric is that the maximal distance between the space points is very small, namely, $n-1$. This means that for a chain metric it is impossible to use an optimum-seeking method with an efficient convergence. Golenko (Ginzburg) [2,3] suggested two other, more efficient metrics which were applied to various scheduling problems. These metrics—the lexicographical metric and the inverse one—will be presented in this paper. The main purpose of this study is to present the essence of our previous papers [2,3] to acquaint the broad mathematical community with our results obtained in this area.

NOTATIONS

D	space of permutations of $\{1,\dots,n\}$;
$\pi \in D$	element of the space;
$\rho(\pi_1,\pi_2)$	numerical function which is defined as the distance between elements $\pi_1,\pi_2 \in D$;
$\rho_{\max}(M)$	maximal distance for the metric M;
$U_R(\pi) = \{\pi_i : \rho(\pi_i,\pi) \le R\}$	R-neighbourhood of $\pi \in D$, $R \in \{1,2,\dots\}$;
$U_r(\pi) = \{\pi_i : \rho(\pi_i,\pi) = r\}$	r-circumference with center $\pi \in D$, $r \in \{1,2,\dots\}$.

LEXICOGRAPHICAL METRIC M_2

In metric M_2, $\rho(\pi_1,\pi_2) = |N(\pi_2) - N(\pi_1)|$ where $N(\pi)$ is an integer function determined on D with the values of the natural row from 1 to $n!$.

DEFINITION OF $N(\pi)$. *Let there be two different permutations*

$$\pi_1 = (\alpha_1, \dots, \alpha_n), \quad \pi_2 = (\beta_1, \dots, \beta_n).$$

Introduce the relation of order into the permutation set as follows: $\pi_1 < \pi_2$ if there exists such an $i\,(1 \le i \le n)$ that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2, \dots, \alpha_{i-1} = \beta_{i-1}$, but $\alpha_i < \beta_i$. We can show that this relation has the property of transitivity, and thus the whole permutation set can be ordered uniquely. Each permutation π is put in correspondence with number $N(\pi)$, defined as the number of the place occupied by permutation π when the whole permutation set is ordered lexicographically.

For each $\pi = (\alpha_1, \dots, \alpha_n) \in D$, $N(\pi)$ satisfies

$$N(\pi) = \sum_{k=1}^{n-1} (l_k - 1)(n-k)! + 1, \tag{1}$$

where l_k is the ordinal number of α_k in the ascending row $1, 2, \dots, n$, values $\alpha_1, \dots, \alpha_{k-1}$ being previously crossed out of the row. Papers [2,3] present relations determining values $\alpha_1, \dots, \alpha_n$ on the basis of $N(\pi)$, and vice versa.

Since for the metric M_2 relation $\rho_{\max}(M_2) = n! - 1 \ge n - 1$ holds, the lexicographical metric is more efficient than the chain one.

INVERSE METRIC M_3

Let $\pi_1 = (\alpha_1, \dots, \alpha_n)$, $\pi_2 = (\beta_1, \dots, \beta_n)$ be two arbitrary elements of permutation space D. If a pair of elements β_i, β_j, $i < j$, is found in π_2, such that relations $\alpha_k = \beta_i$, $\alpha_l = \beta_j$, $k > l$ hold, we will say that pair β_i, β_j forms an inversion relative to π_1. The total number of inversions of π_2 relative to π_1 is defined as the distance $\rho(\pi_1, \pi_2)$, with $\rho_{\max}(M_3) = n(n-1)/2$.

THEOREM. *Each $U_r(\pi)$, $\pi \in D$, with metric M_3 contains as many permutations as there exist different representations of number r as sum*

$$r = \sum_{i=1}^{n-1} \alpha_i, \quad 0 \le \alpha_i \le n - i, \tag{2}$$

taking into account the order of the summands.

In papers [2,3] we present algorithms to simulate a random permutation uniformly distributed both in $U_R(\pi)$ and $U_r(\pi)$ for an arbitrary $\pi \in D$ with metric M_3.

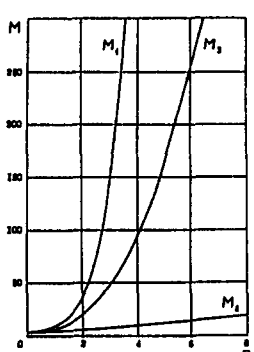

Figure 1. The comparative efficiency of the metrics.

Comparative Efficiency Of The Metrics

Since relation $\rho_{\max}(M_1) \le \rho_{\max}(M_3) \le \rho_{\max}(M_2)$ holds, metric M_2 can be considered as the most efficient one. Note, that the efficiency of the metric space depends also on the problem for the solving of which the permutation space is used. Figure 1 provides the average number of feasible schedules M in $U_R(\pi)$ for a general job-shop scheduling problem with n jobs and m machines, using metrics M_1, M_2 and M_3. The experimentation was carried out for the case $m = n = 6$ and $m = n = 10$, with the initial data matrices being taken from Ref. [4]. A conclusion can be drawn from Figure 1 that the least efficient metric is M_1 and the most efficient one is metric M_2.

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